TMA 01 (Part 2)

In Proclus' commentary on Proposition 20 of Book I of Euclid's *Elements*, explain the argument of 'the Epicureans', and Proclus' rejoinder. What can we learn from this passage about Greek attitudes to proof?

Proclus' commentary on Euclid's Proposition 20 of his first book is important for it clearly shows that during the fifth and fourth centuries B.C. there existed contrasting schools of thought which were concerned with the status of different kinds of knowledge. The distinction between the knowledge acquired through the senses and that attained through reason alone became a central issue amongst Greek philosophers and mathematicians.

Euclid's Proposition 20 states that *in any triangle, two sides taken together in any manner are greater than the remaining one*, and thus would seem to be an obvious statement, which, according to Proclus, the Epicureans were keen to ridicule. This group of thinkers belonged to the critics of geometry and it is interesting to look briefly at the characteristics of each 'faction' within this movement.

Through Proclus we learn that the **Sceptics** "would do away with all knowledge, like enemy troops destroying the crops of a foreign country, in this case a country that has produced philosophy" (**SB** 5.B2) and thus seem to be the most radical of all. The **Epicureans**, on the other hand, strongly questioned only the principles of geometry and it is clearly this aspect which prompted them to criticise Euclid in the passage mentioned above. Finally (and very interestingly) Proclus reports about **Zeno of Sidon** and his followers, who, despite belonging to the Epicureans, took a slightly different stance maintaining that the propositions coming after the principles can be only demonstrated provided they do not grant something that the principles themselves have not previously formulated.[1]

Proclus responds to the critics by saying that it is the task of science to explain even what is clear to perception. To reinforce his argument he mentions two aspects of daily life, such as the fact that fire warms and that humans can move from one point to another. Such occurrences are obvious to anyone, however, the underlying processes ought to be explained in scientific terms. As someone who, at some stage, has been the head of the Neoplatonic Academy himself, Proclus can be forgiven for adopting a platonic thinking and for counter-attacking the Epicureans' criticism.

In this period of time, the attitude towards proof in Greece was particularly subject to division, with the **Eleatics** putting more emphasis on logical considerations (which culminated in their influential proof style called *reductio ad absurdum*), the **Pythagoreans** on religious and ethical ones, while the **Sophists** being more argumentative and rhetorical in their approach. In contrast to this, Socrates and Plato

1. This particular reminds me of the first incompleteness theorem by the twentieth century mathematical logician Kurt Gödel's which states that in any axiomatic mathematical system that includes arithmetic, there will always be statements that are neither provable nor disprovable within the system; i.e., it is impossible to prove within the system all the truths about the system.

made often use of *diknume* proofs, presenting truths in such a way that they can be literally ‘seen’ and which, to my mind, form the basis for proofs by constructions, developed to provide more logical arguments than *diknume* and deeper understanding than *reductio ad absurdum*. The effectiveness of this type of proof lies in the way the initially given mathematical object is modified by construction into one enabling the truth of the proposition to be seen more easily (and this is exactly what Euclid uses frequently in his *Elements*). However, proofs by construction were also meant to have a far-lasting repercussion on mathematical thinking of the subsequent centuries: the idea of finding out what could be constructed by using only a straight-edge and pair of compasses would ultimately lead to the three famous classical problems of ancient Greek mathematics.

Choose one of the three classical problems of Greek mathematics. Explain what it is, and why it was important in the development of mathematics during Greek times.

For my brief essay on the significance of the three classical problems of Greek mathematics, I have chosen the one concerned with doubling the cube:

> *Given a cube, the problem consists in constructing another of twice its volume, i.e., if* $\boldsymbol{s}$ *is the side of the given cube (with volume* $\boldsymbol{s}^3$*), to construct a line, the cube on which would have volume* $2\boldsymbol{s}^3$*.*

The first observation to make is that, like the other two classical problems, doubling the cube requires to be solved by means only of the fundamental constructions (using an unmarked straight-edge and compasses). And, although one might be tempted to suggest doubling the side in order to achieve the result, it takes a simple arithmetical calculation to reveal that doing this would produce a cube having a volume *eight* times that of the original.

The second point of interest is related to the concept of *reduction*. Once more, Proclus can enlighten us here when he writes: “Reduction is a transition from a problem or a theorem to another which, if known or constructed, will make the original proposition evident.” (**SB** 2.F2). We can trace back to Hippocrates the idea of linking the solution of the second classical problem to one of the fundamental concepts in Greek geometry: that of *mean proportional* [2]. The mathematician from Chios established that if two mean proportionals could be found between two lines $\boldsymbol{s}$ and $\mathbf{2}\boldsymbol{s}$, the cube would be doubled:

- the two mean proportionals between s and $2s$ can be written as $s{:}x = x{:}y = y{:}2s$, or $\frac{s}{x} = \frac{x}{y} = \frac{y}{2s}$. So, $\left(\frac{s}{x}\right)^3 = \frac{s}{x}\cdot\frac{x}{y}\cdot\frac{y}{2s}$ whose value is $\frac{1}{2}$.
- it follows that $\left(\frac{s}{x}\right)^3 = \frac{1}{2}$ and finally $x^3 = 2s^3$.

2. The mean proportional between two lines a and b is a line x with the property that $a{:}x = x{:}b$ or, in another form: $\frac{a}{x} = \frac{x}{b}$, which is equivalent to $x = \sqrt{ab}$ (known today as the *geometric mean* between two numbers). In his Elements II, 14, Euclid proved that finding such a line can be achieved by only using an unmarked straight-edge and compasses.

Finding two mean proportionals turned out to be impossible to solve by straightedge and compasses, and it is this important fact that challenged the most renowned mathematicians of ancient Greece. As a result, several solutions were devised; some of them using a *neusis* [3], some others based on complicated constructions. However, one of the most significant approaches is attributed to an associate of Plato, Menaechmus, who, probably for the first time, took as basis for his research the conic sections, generated by intersecting a cone with a surface.

Starting from the original statement $a:x = x:y = y:b$, Menaechmus constructed a system of three equations:

(1) $x^2 = ay$

(2) $y^2 = bx$

(3) $xy = ab$

From these it became clear that curves could be produced by assigning various values for the variables and drawing lines to join them together. It is possible that in this way some fundamental properties of the parabola and hyperbola were discovered.

Overall, the general consensus on the problem of doubling the cube is that it has enabled mathematicians and geometers of ancient Greece to widen their perspective on the relation between practical problems (such as doubling an altar or a tomb) and some intrinsic characteristics of solid figures.

References

FAUVEL, J. (1987) 'The Greek Concept of Proof' part of *Mathematics in the ancient world,* MA290: Topics in the History of Mathematics, The Open University.

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3. Neusis consists of inserting a line of pre-determined length between two other lines (either straight or curved) in such a way that it points in a given direction, verging towards a fixed point. Along with line-and-circle constructions this was an additional method used by ancient Greek mathematician to solve problems which otherwise would have been impossible to prove.