MA290 — TMA 04

Describe the following two mathematical developments of the nineteenth-century and explain in what respects they do, or do not, support the view that nineteenth-century research in mathematics was largely conducted for its own sake: Galois Group Theory & Cantor on Set Theory.

When assessing the numerous mathematical developments of the past centuries, it is not always easy to determine whether they happened as a consequence to outside changes in society, or whether they represented an intrinsic (and often individual) need to expand human knowledge.

The two important theories discussed here have both had repercussions on later mathematics of the twentieth century up to the present times.

Galois' theory of Groups

The young French mathematician Evariste Galois is certainly a unique figure in the history of mathematics. Directly involved in political movements during the July revolution of 1830, his life ended tragically at the early age of twenty in a duel.

The theory of groups has its origins in Galois' interest in the area of polynomial equations, studied thoroughly by Lagrange from the 1770's onwards. For centuries it had been known that only the solutions to quadratic, cubic and quartic equations could be expressed in terms of their coefficients[1]. However, nobody had been able to fully understand why the mechanism did not work for quintic and higher-grade equations, nor to prove that this indeed was impossible.

A first important milestone was provided by Lagrange when he developed the idea of *resolvent*: an expression in the roots of the original equation which depended, in a particular way, on the disposition of the roots themselves. By interchanging the various roots, Lagrange had been able to observe that some polynomials altered their value, while others remained the same.

Starting from this idea, the French mathematician Augustin Cauchy conjectured that the answer to the question of solvability might actually lie in the resolvents, hence his suggestion that careful study should be devoted to them. Suggestion which was taken up by many famous scholars of the nineteenth century; amongst others by Paolo Ruffini, Nils Abel, and, of course, by Carl Friedrich Gauss.

However, it was Galois who, having become familiar with Couchy's study of permutations, had the idea of correlating the set of permutations of the roots of a given equation with its solvability. He observed that, when this set had a cer-

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1. For example, the general quadratic equation $ax^2 + bx + c = 0$ has the solutions $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. This method is called *solvability by radicals*, and involves only the four basic operations and root extraction.

tain structure, it was possible to say whether or not the equation was solvable by radicals. Galois called such sets *groups*[2].

Thanks to his thorough analysis of groups, the young French mathematician succeeded in providing a powerful tool for a deeper understanding of the nature of polynomials equations. However, his theory turned out to have also important, and more far-reaching applications in many other areas of mathematics: Olinde Rodrigues used it to study the composition of motion, Camille Jordan to codify the laws underlying the formation of crystals, and Felix Klein applied it to his research on geometry. In the present century Group theory has become one of the salient mathematical ideas at the centre of the study of differential equations and particle physics. Even in very recent times, Evariste Galois has served the mathematical community when the British scholar Andrew Wiles based his proof of the Taniyama-Shimura conjecture on Group theory. This was to become Wiles' first step towards solving that most notorious of all mathematical riddles: Fermat's Last Theorem, which he finally proved in 1993.

Conclusion

With these considerations in mind I would be inclined to say that the development of Group theory was due, at least initially, to a quite abstract "obsession" of the mathematical community: unlocking the secret of polynomial equations and the laws which governed their roots. Thus, it must belong to that section of scientific research which has served more the spirit and intellect of those involved than society in general.

Cantor's Set theory

During the long history of mathematics, from the times of the Greek philosophers to the modern times, one of the most daunting concepts mathematicians had to confront was that of *infinite*. Some of the oddities associated with infinite quantities have been known since Galileo, but it was the German mathematician Georg Cantor who studied them systematically and, through his efforts, made *set theory* the common language of abstract mathematics[3].

Cantor published his theory for the first time in the 1870's and immediately highlighted the difficulty in defining, indeed understanding, the notion of an infinite set of objects. The question of whether some infinite sets could be larger than other infinite ones was crucial to his study. Cantor introduced the differentiation between *countability* and *uncountability*, along with the notion of *cardinal number* of a set. These new concepts were applied in particular to the analysis of the numeric system, that is, whether integers, rational fractions, algebraic irrational and non-algebraic irrational numbers formed countable or uncountable infinite sets[4]. Thus, in 1895, the German mathematician published his

2. The modern definition of *group* is a set of elements which can be combined together using some operation, such as addition or multiplication, and which satisfy certain conditions. The fundamental property of a *group* is that, when any of two of its elements are combined using the operation, the result is another element of the group.
3. A set in this context is any collection of objects, both concrete or abstract with each item being an *element* of the set.
4. A set is countably infinite is there is some way to associate its elements (with not leftovers) in a one-to-one correspondence with the positive whole numbers. An uncountably infinite set is one whose elements cannot be so matched up with the positive integers.

renowned proof that the rationals are countable which is today regarded as one of the most beautiful contributions to human knowledge, in its simplicity, yet indisputability.

Among the open questions left by the set theory is the so-called *continuum hypothesis*: Cantor speculated that there was no subset of real numbers which was more numerous than the integers, yet less numerous than the real numbers, and so far no-one has been able to prove whether this is true or not. Indeed, this might turn out to be an impossible task, after some 20th-century logicians showed that the hypothesis is independent of the other axioms of set theory: both affirmative and negative answers to the question are consistent with the present understanding of sets.

The influence of set theory cannot be overstated; a whole generation of scholars pursued the issue of one-to-one correspondence, starting from Cantor's and Dedekind's research on equivalences between the set of all points of a line and the set of all points of two-dimensional space, to Giuseppe Peano's startling space-filling curve.

More importantly, the theory forced a re-thinking of the foundations of mathematics itself. Both Cantor and Gottlob Frege, for example, shared a hostility to the idea that mathematics was about emotions, feelings, sensations, or physical perceptions. Logic, they argued, could be used to derive all of arithmetic and the notion of numbers, although not geometry, which was more closely connected to the real world. The British logician and mathematician Bertrand Russell went even further, proving that a small number of fundamental logical principles can serve as basis for constructing the entire edifice of pure mathematics. The whole subsequent debate of *formalism* vs. *intuitionism*[5] as well as the discovery of paradoxes in the theory are also a direct consequence of Cantor's impetus.

Conclusion

Considering that the study of mathematical sciences has seen immense advances since the publication of set theory, it is not difficult to recognise that Cantor's work, despite being extremely abstract in contents, has, possibly for the first time, allowed the scientific community to reflect more radically on the meaning of those very mathematical constructs which, in large part, are an extension of the human intellect. Its impact continues to be relevant to many disciplines, including physics, mathematics, logic, and philosophy.

References

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5. *Formalism* stresses the importance of laws governing mathematics and establishes this science on a strictly axiomatic base; on the other hand, *intuitionism* focuses more on the ability of the human mind to make use of intuition and creativity, in order to acknowledge truths.